10 Orthogonal Vectors

Orthogonality In an inner-product space $\mathcal{V}$, two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ are said to be orthogonal (to each other) whenever $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, and this is denoted by writing $\boldsymbol{x} \perp \boldsymbol{y}$.

- For $\mathbb{R}^n$ with the standard inner product, $\boldsymbol{x} \perp \boldsymbol{y} \iff \boldsymbol{x}^\top \boldsymbol{y} = 0$.
- For $\mathbb{C}^n$ with the standard inner product, $\boldsymbol{x} \perp \boldsymbol{y} \iff \boldsymbol{x}^* \boldsymbol{y} = 0$.

1. Using the standard inner product, determine which of the following pairs are orthogonal vectors in the indicated space. (a) $x = (1, -3, 4)^\top$ and $y = (-2, 2, 2)^\top$ in $\mathbb{R}^3$. (b) $x = (i, 1+i, 2, 1-i)^\top$ and $y = (0, 1+i, -2, 1-i)^\top$ in $\mathbb{C}^4$. (c) $x = (1, -2, 3, 4)^\top$ and $y = (4, 2, -1, 1)^\top$ in $\mathbb{R}^4$. (d) $x = (1+i, 1, i)^\top$ and $y = (1-i, -3, -i)^\top$ in $\mathbb{C}^3$. (e) $x = (0, 0, ..., 0)^\top$ and $y = (y_1, y_2, ..., y_n)^\top$ in $\mathbb{R}^n$.

Angles In a real inner-product space $\mathcal{V}$, the radian measure of the angle between nonzero vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ is defined to be the number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|}.$$

Orthonormal Sets $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ is called an orthonormal set whenever $\|\boldsymbol{u}_i\| = 1$ for each i, and $\boldsymbol{u}_i \perp \boldsymbol{u}_j$ for all $i \neq j$. In other words,

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

- Every orthonormal set is linearly independent.
- Every orthonormal set of n vectors from an n-dimensional space $\mathcal{V}$ is an orthonormal basis for $\mathcal{V}$.

2. (a) Check is it true that $x = (1, -2, 3, -1)^\top$ is orthogonal to $y = (4, 1, -2, -4)^\top$. (b) Determine the angle between $x = (-4, 2, 1, 2)^\top$ and $y = (1, 0, 2, 2)^\top$.

3. Find two vectors of unit norm that are orthogonal to $\boldsymbol{u} = (3, -2)^\top$.

4. Consider the following set of three vectors $\mathcal{B}' = \{\boldsymbol{u}_1 = (1, -1, 0)^\top, \boldsymbol{u}_2 = (1, 1, 1)^\top, \boldsymbol{u}_3 = (-1, -1, 2)^\top\}$. (a) Using the standard inner product in $\mathbb{R}^3$, verify that these vectors are mutually orthogonal. (b) Convert the set $\mathcal{B}'$ into an orthonormal basis for $\mathbb{R}^3$.

5. Consider the following set of three vectors

$$\left\{ x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

(a) Using the standard inner product in $\mathbb{R}^4$ check are these vectors orthogonal between themselves. (b) Find a nonzero vector x_4 such that $\{x_1, x_2, x_3, x_4\}$ is a set of mutually orthogonal vectors. (c) Convert the resulting set into an orthonormal basis for $\mathbb{R}^4$.

Fourier[8] **expansion** of $\boldsymbol{x} = \xi_1 \boldsymbol{u}_1 + \xi_2 \boldsymbol{u}_2 + ... + \xi_n \boldsymbol{u}_n$.

Fourier Expansions If $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ is an orthonormal basis for an inner-product space $\mathcal{V}$, then each $\boldsymbol{x} \in \mathcal{V}$ can be expressed as

$$\boldsymbol{x} = \langle \boldsymbol{u}_1, \boldsymbol{x} \rangle \boldsymbol{u}_1 + \langle \boldsymbol{u}_2, \boldsymbol{x} \rangle \boldsymbol{u}_2 + ... + \langle \boldsymbol{u}_n, \boldsymbol{x} \rangle \boldsymbol{u}_n. \quad (1)$$

This is called the Fourier expansion of x. The scalars $\xi_i = \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle$ are the coordinates of $\boldsymbol{x}$ with respect to $\mathcal{B}$, and they are called the Fourier coefficients. Geometrically, the Fourier expansion resolves $\boldsymbol{x}$ into n mutually orthogonal vectors $\langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \boldsymbol{u}_i$, each of which represents the orthogonal projection of $\boldsymbol{x}$ onto the space (line) spanned by $\boldsymbol{u}_i$.

6. Determine the Fourier expansion of $\boldsymbol{x} = (-1, 2, 1)^\top$, with respect to the standard inner product and the orthonormal basis given in Exercise 4, $\mathcal{B} = \{\boldsymbol{u}_1 = \frac{1}{\sqrt{2}}(1, -1, 0)^\top, \boldsymbol{u}_2 = \frac{1}{\sqrt{3}}(1, 1, 1)^\top, \boldsymbol{u}_3 = \frac{1}{\sqrt{6}}(-1, -1, 2)^\top\}$.

7. Given an orthonormal basis $\mathcal{B}$ for a space $\mathcal{V}$, explain why the Fourier expansion for $x \in \mathcal{V}$ is uniquely determined by $\mathcal{B}$.

InC: 1, 3, 5, 6. HW: see `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.

[8] Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician and physicist who, while studying heat flow, developed expansions similar to (1). Fourier's work dealt with special infinite-dimensional inner-product spaces involving trigonometric functions as discussed in HW. Although they were apparently used earlier by Daniel Bernoulli (1700–1782) to solve problems concerned with vibrating strings, these orthogonal expansions became known as Fourier series, and they are now a fundamental tool in applied mathematics. Born the son of a tailor, Fourier was orphaned at the age of eight. Although he showed a great aptitude for mathematics at an early age, he was denied his dream of entering the French artillery because of his "low birth." Instead, he trained for the priesthood, but he never took his vows. However, his talents did not go unrecognised, and he later became a favorite of Napoleon. Fourier's work is now considered as marking an epoch in the history of both pure and applied mathematics. The next time you are in Paris, check out Fourier's plaque on the first level of the Eiffel Tower.

11 Gram-Schmidt Procedure

Let $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}2, ..., \boldsymbol{x}_n\}$ be an arbitrary basis (not necessarily orthonormal) for an n-dimensional inner-product space $\mathcal{S}$, and remember that $\| \star \| = \langle \star, \star \rangle^{1/2}$. **Objective:** Use $\mathcal{B}$ to construct an orthonormal basis $\boldsymbol{O} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ for $\mathcal{S}$. **Strategy:** Construct $\boldsymbol{O}$ sequentially so that $\boldsymbol{O}_k = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$ is an orthonormal basis for $\mathcal{S}_k = \text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_k\}$ for $k = 1, ..., n$.

Gram-Schmidt Orthogonalization Procedure If $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ is a basis for a general inner-product space $\mathcal{S}$, then the Gram-Schmidt sequence defined by $\boldsymbol{u}_1 = \dfrac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$ and

$$\boldsymbol{u}_k = \frac{x_k - \sum_{i=1}^{k-1} \langle \boldsymbol{u}_i, \boldsymbol{x}_k \rangle \boldsymbol{u}_i}{\|x_k - \sum_{i=1}^{k-1} \langle \boldsymbol{u}_i, \boldsymbol{x}_k \rangle \boldsymbol{u}_i\|} \text{ for } k = 2, ..., n$$

is an orthonormal basis for $\mathcal{S}$. When $\mathcal{S}$ is an n-dimensional subspace of $\mathbb{C}^m$, the Gram-Schmidt sequence can be expressed as

$$\boldsymbol{u}_k = \frac{(I - U_k U_k^*)\boldsymbol{x}_k}{\|(I - U_k U_k^*)\boldsymbol{x}_k\|} \quad \text{for} \quad k = 1, 2, ..., n$$

in which $U_1 = \boldsymbol{0} \in \mathbb{C}^m$ and $U_k = (\boldsymbol{u}_1 | \boldsymbol{u}_2 | ... | \boldsymbol{u}_{k-1})_{m \times k-1}$ for $k > 1$.

Classical Gram-Schmidt Algorithm. The following formal algorithm is the straightforward or "classical" implementation of the Gram-Schmidt procedure. Interpret $a \leftarrow b$ to mean that "a is defined to be (or overwritten by) b."

For $k = 1$: $\boldsymbol{u}_1 \leftarrow \dfrac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$

For $k > 1$: $\boldsymbol{u}_k \leftarrow \boldsymbol{x}_k - \displaystyle\sum_{i=1}^{k-1} (\boldsymbol{u}_i^* \boldsymbol{x}_k) \boldsymbol{u}_i$, $\boldsymbol{u}_k \leftarrow \dfrac{\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|}$.

1. Use the classical formulation of the Gram-Schmidt[9] procedure given above to find an orthonormal basis for the space spanned by the following three linearly independent vectors: $\boldsymbol{x}_1 = (1, 0, 0, -1)^\top$, $\boldsymbol{x}_2 = (1, 2, 0, -1)^\top$, $\boldsymbol{x}_3 = (3, 1, 1, -1)^\top$.

2. Use the variation of Gram-Schmidt algorithm and in 4-dimensional inner-product space $\mathbb{R}^4$ find an orthonormal basis for the space spanned by the following set of vectors

$$\{(1, 2, 0, 1)^\top, (0, 2, 0, 2)^\top, (1, -1, 1, 1)^\top, (1, 0, 0, 0)^\top\}.$$

QR Factorization Every matrix $A_{m \times n}$ with linearly independent columns can be uniquely factored as $A = QR$ in which the columns of $Q_{m \times n}$ are an orthonormal basis for $\text{im}(A)$ and $R_{n \times n}$ is an upper-triangular matrix with positive diagonal entries.

- The QR factorization is the complete "road map" of the Gram-Schmidt process because the columns of $Q = (\boldsymbol{q}_1 | \boldsymbol{q}_2 | ... | \boldsymbol{q}_n)$ are the result of applying the Gram-Schmidt procedure to the columns of $A = (\boldsymbol{a}_1 | \boldsymbol{a}_2 | ... | \boldsymbol{a}_n)$ and R is given by

$$R = \begin{bmatrix} \nu_1 & \boldsymbol{q}_1^* \boldsymbol{a}_2 & \boldsymbol{q}_1^* \boldsymbol{a}_3 & ... & \boldsymbol{q}_1^* \boldsymbol{a}_n \\ 0 & \nu_2 & \boldsymbol{q}_2^* \boldsymbol{a}_3 & ... & \boldsymbol{q}_2^* \boldsymbol{a}_n \\ 0 & 0 & \nu_3 & ... & \boldsymbol{q}_3^* \boldsymbol{a}_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & ... & \nu_n \end{bmatrix}$$

where $\nu_1 = \|\boldsymbol{a}_1\|$ and $\nu_k = \|\boldsymbol{a}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{q}_i, \boldsymbol{a}_k \rangle \boldsymbol{q}_i\|$ for $k > 1$.

3. Determine the QR factors of

$$A = \begin{pmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{pmatrix}.$$

Variation of Gram-Schmidt Algorithm. For a linearly independent set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$:

for $k = 1$: $\boldsymbol{u}_1 \leftarrow \boldsymbol{x}_1$

for $k > 1$: $\boldsymbol{u}_k \leftarrow \boldsymbol{x}_k - \displaystyle\sum_{i=1}^{k-1} \frac{\langle \boldsymbol{u}_i, \boldsymbol{x}_k \rangle}{\|\boldsymbol{u}_i\|^2} \boldsymbol{u}_i$

for $k \geq 1$: $\boldsymbol{u}_k \leftarrow \dfrac{\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|}$

4. Let $T : \text{Mat}_{n \times n}(\mathbb{R}) \to \mathcal{P}_3$ be a given linear transformation defined with $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (3a + 2c - d)x^3 + (3b - c - d)x^2 + (2a + b + c - d)x + (a + 2b - d)$. Find an orthonormal basis for $\text{im}(T)$. (Use standard inner product for $\mathcal{P}_3$. Recall: $\langle a_3 x^3 + a_2 x^2 + a_1 x + a_0, b_3 x^3 + b_2 x^2 + b_1 x + b_0 \rangle = a_3 b_3 + a_2 b_2 + a_1 b_1 + a_0 b_0$).

[9] Jorgen P. Gram (1850–1916) was a Danish actuary who implicitly presented the essence of orthogonalization procedure in 1883. Gram was apparently unaware that Pierre-Simon Laplace (1749–1827) had earlier used the method. Today, Gram is remembered primarily for his development of this process, but in earlier times his name was also associated with the matrix product A^*A that historically was referred to as the *Gram matrix* of A.

Erhard Schmidt (1876–1959) was a student of Hermann Schwarz (of CBS inequality fame) and the great German mathematician David Hilbert. Schmidt explicitly employed the orthogonalization process in 1907 in his study of integral equations, which in turn led to the development of what are now called Hilbert spaces. Schmidt made significant use of the orthogonalization process to develop the geometry of Hilbert Spaces, and thus it came to bear Schmidt's name.